where f is any arbitrary function

TYPE –III

In the next example, we find the solution of $Pp + Qq = R$ ...(*i*)

by the following formula (from algebra) *i.e.*,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{S\,dx + T\,dy + U\,dz}{PS + QT + RU}$$

where S, T, U are some functions of x, y, z

If S, T, U are chosen in such a manner that (known as multipliers) $PS + QT + RU = 0$

Then we have $S\,dx + T\,dy + U\,dz = 0$

Now integrate it to get one independent solution of (*i*) as $u(x, y, z) = a$.

And the other independent solution can obtained either by selecting another set of multipliers or by taking two members of auxiliary equations.

Example 3. Find the general solution of following lagrange's linear equations

(*i*) $(z-y)p + (x-z)q = y-x$

(*ii*) $z(xp - qy) = y^2 - x^2$

(*iii*) $(y^2 + z^2)p - xyq = -zx$

(*iv*) $x(y-z)p + y(z-x)q = z(x-y)$

(*v*) $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

(*vi*) $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$

(*vii*) $(y + xz)p - (x + yz)q = x^2 - y^2$

(*viii*) $\left(\frac{1}{z} - \frac{1}{y}\right)p + \left(\frac{1}{x} - \frac{1}{z}\right)q = \frac{1}{y} - \frac{1}{x}$

(*ix*) $(3z - 4y)p + (2z - 4x)q = 3x - 2y$

(*x*) $(\beta z - \gamma y)p + (\gamma x - \alpha z)q = \alpha y - \beta x$.

Sol. (*i*) We are given the differential equation

$$(z-y)p + (x-z)q = y-x$$

Compare it with $Pp + Qq = R$

Here $P = z - y$, $Q = x - z$, $R = y - x$

$\therefore$ The auxiliary equations are

$$\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x} \quad ...(i)$$

Taking multipliers as 1, 1, 1 ; each of fraction of (*i*)

$$= \frac{dx + dy + dz}{1(z-y) + 1(x-z) + 1(y-x)} = \frac{dx + dy + dz}{0}$$

$\therefore$ $dx + dy + dz = 0$

Integrating, $x + y + z = a$ ...(*ii*)

Further taking multipliers as x, y, z ; each of

$$\text{fraction of } (i) = \frac{x\,dx + y\,dy + z\,dz}{x(z-y) + y(x-z) + z(y-x)} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\therefore$ $x\,dx + y\,dy + z\,dz = 0 \Rightarrow 2x\,dx + 2y\,dy + 2z\,dz = 0$

Integrating $x^2 + y^2 + z^2 = b$ ...(*iii*)

From (*ii*) and (*iii*), we get $u = a$ and $v = b$

where $u(x, y, z) = x + y + z$, $v(x, y, z)\ x^2 + y^2 + z^2$

$\therefore$ The general solution is given by

$f(x + y + z, x^2 + y^2 + z^2) = 0$

where f is any arbitrary function.

(*ii*) We are given differential equation

$$z(xp - qy) = y^2 - x^2 \quad \text{or} \quad zxp - zyq = y^2 - x^2$$

Compare it with $Pp + Qq = R$ where

$$P = zx\ ;\ Q = -zy\ ;\ R = y^2 - x^2$$

The auxiliary equations are

$$\frac{dx}{zx} = \frac{dy}{-zy} = \frac{dz}{y^2 - x^2} \quad ...(i)$$

Taking multipliers as x, y, z ; each of fraction of (*i*)

$$= \frac{x\,dx + y\,dy + z\,dz}{zx^2 - zy^2 + z(y^2 - x^2)} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\therefore$ $x\,dx + y\,dy + z\,dz = 0$

$\Rightarrow$ $2x\,dx + 2y\,dy + 2z\,dz = 0$

Integrating, $x^2 + y^2 + z^2 = a$ ...(*ii*)

Taking first two members of (*i*)

we get
$$\frac{dx}{zx} = -\frac{dy}{zy}$$

or
$$\frac{dx}{x} + \frac{dy}{y} = 0$$

Integrating, $\log|x| + \log|y| = \log|c|$

$\log|x||y| = \log|c| \Rightarrow |x||y| = |c|$ or $|xy| = (c)$

$\Rightarrow$ $xy = \pm c = b$

$\Rightarrow$ $xy = b$ ...(*iii*)

$\Rightarrow$

From (*ii*) and (*iii*), we have $u = a$ and $v = b$

where $u(x, y, z) = x^2 + y^2 + z^2$ and $v(x, y, z) = xy$

$\therefore$ The general sol is given by
$$f(x^2 + y^2 + z^2, xy) = 0$$

where f is any arbitrary function

(*iii*) We are given the differential equation
$$(y^2 + z^2)p - xyq = -zx$$

Compare it with $Pp + Qq = R$

Here $P = y^2 + z^2$ and $Q = -xy$, $R = -zx$

The auxiliary equation are ...(*i*)
$$\frac{dx}{y^2 + z^2} = \frac{dy}{-xy} = \frac{dz}{-zx}$$

Taking x, y, z as multipliers ; each of fraction

of (*i*) $$\frac{x\,dx + y\,dy + z\,dz}{x(y^2 + z^2) - xy^2 - z^2x} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\therefore$ $x\,dx + y\,dy + z\,dz = 0 \Rightarrow 2x\,dx + 2y\,dy + 2z\,dz = 0$ ...(*ii*)

Integrating $x^2 + y^2 + z^2 = a$

Taking last two members of (*i*), we get
$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating $\log|y| = |\log|z| + \log|c|$

$\Rightarrow$ $\log|y| = \log|z||c| \Rightarrow |y| = |z||c| = |cz|$

$\Rightarrow$ $y = \pm cz = bz$ (say) ...(*iii*)

$\Rightarrow$ $\frac{y}{z} = b$

From (*ii*) and (*iii*) we get $u = a$ and $v = b$

where $u(x, y, z) = x^2 + y^2 + z^2$ and $v(x, y, z) = \dfrac{y}{z}$

$\therefore$ The general sol of given equation is

$$f\left(x^2 + y^2 + z^2, \frac{y}{z}\right) = 0.$$

(iv) We are given the differential equation

$$x(y - z)p + y(z - x)q = z(x - y)$$

Compare it with $Pp + Qq = R$ where $P = x(y - z)$, $Q = y(z - x)$, $R = z(x - y)$

The auxiliary equation are

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} \quad \text{...(i)}$$

Taking 1, 1, 1 as multipliers ; each of fraction of (i)

$$= \frac{dx + dy + dz}{1.x(y-z) + 1.y(z-x) + 1.z(x-y)} = \frac{dx + dy + dz}{0}$$

$\Rightarrow$ $dx + dy + dz = 0$

Integrating $x + y + z = a$ ...(ii)

Taking $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$ as multipliers ; each of fraction of (i)

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{1}{x}x(y-z) + \frac{1}{y}y(z-x) + \frac{1}{z}z(x-y)} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

$$\Rightarrow \quad \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating $\log|x| + \log|y| + \log|z| = \log|c|$

$\Rightarrow$ $\log|x||y||z| = \log|c|$

$\Rightarrow$ $\log|xyz| = \log|c|$

$\Rightarrow$ $xyz = \pm c = b$ say

$\Rightarrow$ $xyz = b$ ...(iii)

From (ii) and (iii), the general sol is given by $f(x + y + z, xyz) = 0$.

(v) We are given the differential equation

$$x^2(y - z)p + y^2(z - x)q = z^2(x - y)$$

Compare it with $Pp + Qq = R$

where $P = x^2(y - z)$, $Q = y^2(z - x)$, $R = z^2(x - y)$

The auxiliary equation s are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

or $$\frac{dx}{x^2(y-z)}=\frac{dy}{y^2(z-x)}=\frac{dz}{z^2(x-y)} \quad ...(i)$$

Taking $\frac{1}{x},\frac{1}{y},\frac{1}{z}$ as multipliers ; each of fraction of (*i*)

$$=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{\frac{1}{x}x^2(y-z)+\frac{1}{y}y^2(z-x)+\frac{1}{z}z^2(x-y)}$$

$$=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{x(y-z)+y(z-x)+z(x-y)}=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$

$$\therefore \quad \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0$$

Integrating $\Rightarrow \log|x|+\log|y|+\log|z|=\log|c|$

$\Rightarrow \log|x||y||z|=\log|c| \Rightarrow \log|xyz|=\log|c|$

$\Rightarrow |xyz|=|c| \Rightarrow xyz=\pm c=a$ say

$\Rightarrow \quad xyz=a \quad ...(ii)$

Taking $\frac{1}{x^2},\frac{1}{y^2},\frac{1}{z^2}$ as multipliers , then each of fraction of (*ii*)

$$=\frac{\frac{1}{x^2}dx+\frac{1}{y^2}dy+\frac{1}{z^2}dz}{\frac{1}{x^2}x^2(y-z)+\frac{1}{y^2}y^2(z-x)+\frac{1}{z^2}z^2(x-y)}=\frac{\frac{1}{x^2}dx+\frac{1}{y^2}dy+\frac{1}{z^2}dz}{0}$$

$$\therefore \quad \frac{1}{x^2}dx+\frac{1}{y^2}dy+\frac{1}{z^2}dz=0$$

Integrating

$$\int x^{-2}dx+\int y^{-2}dy+\int z^{-2}dz=-b \text{ (say)}$$

$$\Rightarrow \quad \frac{x^{-1}}{-1}+\frac{y^{-1}}{-1}+\frac{z^{-1}}{-1}=-b \quad ...(iii)$$

$$\Rightarrow \quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=b$$

From (*ii*) and (*iii*), we $u=a$ and $v=b$

where $u(x\,y, z) = x\,y\,z$ and $v(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$

$\therefore$ The general solution is given by

$$f\left(x\,y\,z, \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0$$

where f is a any arbitrary function.

(*vi*) We are given the differential equation

$$x(y^2 - z^2)\,p + y(z^2 - x^2)\,q = z(x^2 - y^2)$$

Compare it with $Pp + Qq = R$

Here $P = x(y^2 - z^2)$, $Q = y(z^2 - x^2)$, $R = z(x^2 - y^2)$

The auxiliary equations are

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)} \quad \text{...(i)}$$

Taking multipliers as $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$; each of fraction of (*i*)

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{1}{x}x(y^2 - z^2) + \frac{1}{y}y(z^2 - x^2) + \frac{1}{z}z(x^2 - y^2)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2 - z^2 + z^2 - x^2 + x^2 - y^2}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\therefore \quad \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating $\log|x| + \log|y| + \log|z| = \log|c|$

$\Rightarrow \quad \log|x||y||z| = \log|c|$

$\Rightarrow \quad \log|x\,y\,z| = \log|c|$

$\Rightarrow \quad |x\,y\,z| = |c|$ or $x\,y\,z = \pm c = a$ say

$\Rightarrow \quad x\,y\,z = a$ ...(*i*)

Taking multiplies as x, y, z; each of fraction of (*i*)

$$= \frac{x\,dx + y\,dy + z\,dz}{x^2(y^2 - z^2) + y^2(z^2 - x^2) + z^2(x^2 - y^2)} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\therefore \quad x\,dx + y\,dy + z\,dz = 0$

$\Rightarrow \quad 2x\,dx + 2y\,dy + 2z\,dz = 0$

Integrating, we get $x^2 + y^2 + z^2 = b$ ...(*ii*)

From (*i*) and (*ii*), we get $u = a$ and $v = b$

where $u(x, y, z) = x\,y\,z$ and $v(x, y, z) = x^2 + y^2 + z^2$

TYPE –IV

In the next example, we find the solution of $Pp + Qq = R$ by the following form i.e.,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{S\,dx + T\,dy + U\,dz}{PS + QT + RU} \quad ...(1)$$

where S, T, U are some functions of x, y, z If the multipliers S,T,U are selected in such a manner that $S\,dx + T\,dy + U\,dz$ is exact differential of a factor of $PS + QT + RU$.

Then we consider two members of (1) One $\frac{S\,dx + T\,dy + U\,dz}{PS + QT + RU}$ and other suitably chosen to get one independent solution. And the other independent solution can be obtained either by selecting another set of multipliers or by taking two members of auxiliary equations.

Example 4. Solve the following Lagrange's linear equations for their general solution.

(i) $(1 + y)\,p + (1 + x)\,q = z$

(ii) $xz\,p + yz\,q = xy$

(iii) $x\,p + q\,z = -y$

(iv) $(x^2 + y^2)\,p + (2xy)\,q = (x + y)z$

(v) $z\,p + (x + y - z)\,q = -z$

(vi) $y\,p + x\,q = z$

(vii) $p\cos(x + y) + q\sin(x + y) = z$

(viii) $(x^2 - yz)\,p + (y^2 - zx)\,q = z^2 - xy$

(ix) $(y + z)\,p + (z + x)\,q = x + y$

(x) $(y^2 + yz + z^2)\,p + (z^2 + zx + x^2)\,q = (x^2 + xy + y^2)$

Sol. (i) We are given the differential equation

$(1 + y)\,p + (1 + x)\,q = z$

Compare it with $Pp + Qq = R$

Here $P = 1 + y$, $Q = 1 + x$, $R = z$

The auxiliary equations are

$$\frac{dx}{1+y} = \frac{dy}{1+x} = \frac{dz}{z} \quad ...(i)$$

Taking first two members of (i), we get

$$(1 + x)\,dx = (1 + y)\,dy$$

$$\Rightarrow \quad 2(1 + x)\,dx = 2(1 + y)\,dy$$

Integrating, $(1 + x)^2 = (1 + y)^2 + a \quad ...(ii)$

$$\Rightarrow \quad (1 + x)^2 - (1 + y)^2 = a$$

Each of fraction of (i) $= \frac{dx + dy}{1 + y + 1 + x} = \frac{dx + dy}{x + y + 2}$

$$\therefore \quad \frac{dx + dy}{x + y + 2} = \frac{dz}{z} \Rightarrow \frac{d(x + y + 2)}{x + y + 2} = \frac{dz}{z}$$

Integrating $\log|x + y + 2| = \log|z| + \log|c| = \log|z||c| = \log|zc|$

$$\Rightarrow \quad |x + y + z| = |zc|$$

$$\Rightarrow \quad x + y + 2 = \pm cz = bz$$

$$\Rightarrow \quad \frac{x+y+2}{z} = b \qquad \ldots(iii)$$

From (*ii*) and (*iii*), General sol is given by

$$f\left((1+x)^2 - (1+y)^2, \frac{x+y+2}{z}\right) = 0$$

where f is any arbitrary function.

(*ii*) We are given the differential equation

$$xzp + yzq = xy$$

Compare it with $Pp + Qq = R$

Here $P = xz$, $Q = yz$ and $R = xy$

$\therefore$ The auxiliary Equations are

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy} \qquad \ldots(1)$$

Taking multipliers as $\frac{1}{x}, \frac{1}{y}, 0$; each of fraction of (*i*)

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + 0\,dz}{\frac{1}{x}(xz) + \frac{1}{y}(yz) + 0.xy} = \frac{\frac{dx}{x} + \frac{dy}{y}}{2z}$$

$$\therefore \quad \frac{\frac{dx}{x} + \frac{dy}{y}}{2z} = \frac{dz}{xy} \Rightarrow \frac{y\,dx + x\,dy}{2zxy} = \frac{dz}{xy}$$

$$\Rightarrow y\,dx + x\,dy = 2z\,dz$$

$$\Rightarrow \quad d(xy) = d(z^2)$$

Integrating $xy = z^2 + a \qquad \ldots(ii)$

$$\Rightarrow \quad xy - z^2 = a$$

Now taking first two members of (*i*)

we get $\quad \frac{dx}{x} = \frac{dy}{y}$

Integrating, $\log|x| = \log|y| + \log|c|$

$\Rightarrow \quad \log|x| = \log|y||c|$

$\Rightarrow \quad |x| = |y||c| \Rightarrow |x| = |cy|$

$\Rightarrow \quad x = \pm cy$ or $\frac{x}{y} = \pm c = b$ (say) $\qquad \ldots(iii)$

or $\quad \frac{x}{y} = b$

From (*ii*) and (*iii*), we get the general solution

as $f\left(xy - z^2, \frac{x}{y}\right) = 0.$

(*iii*) We are given the differential equation as

$xp + zq = -y$

Compare it with $Pp + Qq = R$

Here $P = x$, $Q = z$ and $R = -y$

$\therefore$ The auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{z} = \frac{dz}{-y} \quad ...(i)$$

From last two members of (*i*), we get

$y\,dy = -z\,dz \quad \Rightarrow \quad 2y\,dy + 2z\,dz = 0$

Integrating $y^2 + z^2 = a \quad ...(ii)$

Now taking multipliers as $0, z, -y$; each member of (*i*)

$$= \frac{0\,dx + z\,dy - y\,dz}{0 + z^2 + y^2} = \frac{z\,dy - y\,dz}{z^2 + y^2} = \frac{\frac{1}{z}dy - \frac{y}{z^2}dz}{1 + \left(\frac{y}{z}\right)^2} = \frac{d\left(\frac{y}{z}\right)}{1 + \left(\frac{y}{z}\right)^2}$$

$\therefore$ we have $\frac{dx}{x} = \frac{d\left(\frac{y}{z}\right)}{1 + \left(\frac{y}{z}\right)^2}$ (Taking first member of (*i*))

Integrating $\log|x| = \tan^{-1}\frac{y}{z} + \log b$

$$\Rightarrow \quad \log|x| - \log b = \tan^{-1}\frac{y}{z}$$

$$\Rightarrow \quad \log\frac{|x|}{b} = \tan^{-1}\frac{y}{z}$$

$$\Rightarrow \quad \frac{|x|}{b} = e^{\tan^{-1}\frac{y}{z}} \Rightarrow \frac{|x|}{e^{\tan^{-1}\frac{y}{z}}} = b \quad ...(iii)$$

From (*ii*) and (*iii*), we get general sol as

$$f\left(y^2 + z^2, \frac{|x|}{e^{\tan^{-1}\frac{y}{z}}}\right) = 0.$$

(*iv*) We are given the differential equation as

$(x^2 + y^2)\,p + 2xyq = (x + y)\,z$

Compare it with $Pp + Qq = R$

Here $P = (x^2 + y^2)$ and $Q = 2xy$, $R = (x + y)z$

$\therefore$ The auxiliary equations are

$$\frac{dx}{x^2 + y^2} = \frac{dy}{2xy} = \frac{dz}{(x+y)z} \qquad \ldots(i)$$

Taking multipliers as 1, 1, 0 ; each of member of (*i*)

$$= \frac{dx + dy + 0\,dz}{1(x^2 + y^2) + (2xy) + 0(x+y)} = \frac{dx + dy}{(x+y)^2} = \frac{d(x+y)}{(x+y)^2} \qquad \ldots(ii)$$

$$\therefore \quad \frac{d(x+y)}{(x+y)^2} = \frac{dz}{(x+y)z}$$

$$\Rightarrow \quad \frac{d(x+y)}{x+y} = \frac{dz}{z}$$

Integrating $\log|x+y| = \log|z| + \log|c|$

$$\Rightarrow \quad \log|x+y| - \log|c| = \log|z|$$

$$\Rightarrow \quad \log\frac{|x+y|}{|c|} = \log|z|$$

$$\Rightarrow \quad \frac{|x+y|}{|c|} = |z| \quad \text{or} \quad \left|\frac{x+y}{z}\right| = |c|$$

$$\Rightarrow \quad \frac{x+y}{z} = \pm c = a \quad \text{say}$$

$$\therefore \quad \frac{x+y}{z} = a \qquad \ldots(iii)$$

Taking multipliers as 1, – 1, 0 ; each of member of (*i*)

$$= \frac{dx - dy + 0\,dz}{1(x^2 + y^2) - 2xy + 0} = \frac{d(x-y)}{(x-y)^2} \qquad \ldots(iv)$$

From (*ii*) and (*iv*)

$$\frac{d(x+y)}{(x+y)^2} = \frac{d(x-y)}{(x-y)^2}$$

Integrating, we get

$$\frac{(x+y)^{-2+1}}{-2+1} = \frac{(x-y)^{-2+1}}{-2+1} + b$$

$$\Rightarrow \quad -\frac{1}{x+y} = -\frac{1}{x-y} + b$$

$$\Rightarrow \quad \frac{1}{x-y} - \frac{1}{x+y} = b$$